Portfolio Evaluation Under Uncertainty

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1 Introduction

Principles for evaluation of the risk associated with a portfolio of investment projects within a company are based on models that describe potential changes in the factors that influence the profit of the individual projects and the interaction between these. Many different models are described in the literature. Some approaches are simple and may border towards the trivial, whilst others are more involved and rely on advanced stochastic models. During the last years it has become popular to apply principles and theory from the mathematics of finance (see e.g. [1] and [2]). Analogy is drawn between investment in securities on the one hand and investment in projects on the other. Both situations are characterised by the investment of an amount of money at one or several points in time with the hope of receiving larger sums of money at one or more future points in time. Such analogy has its strong and weak sides.

In this paper we describe an approach based on what we call *Risk-Adjusted Expected Net Present Value (RAENPV)* and contrast it to the more common *Net Present Value (NPV) with Risk Adjusted Discount Rate*. We also propose a way of describing the uncertainty associated with the cash flow of each individual candidate project in a portfolio and how it interacts with each of the other candidate projects. Our guiding principle has been to establish an approach, which on the one hand is not too narrow but on the other hand avoids relying on complicated models that cannot be justified because of lack of knowledge about the future.

2 Evaluation of a Single Project

2.1 Risk-adjusted Discount Rate

We shall first briefly comment in general terms on an approach which is often used where a project is rated according to its NPV calculated using a risk-adjusted discount rate that reflects the uncertainties in the cash flow. The approach is usually based on variants of the *Capital Asset Pricing Model (CAPM)*. A thorough description of CAPM may be found in [1]. A feature of this approach is that high uncertainty in the cash flow leads to a high risk-adjusted discount rate. The approach is not without problems, even for projects with a lifetime of one period only. We mention some of the conditions that must be satisfied:

- Investors have expectations about asset returns that are normally distributed.
- Investors may borrow or lend unlimited amounts of a risk-free asset.
- All projects (and parts of projects) can be bought/sold in a perfect market.

Even stronger are the conditions that must be satisfied if we want to use a risk-adjusted discount rate for projects with a lifetime of more than one period. Then it must be assumed that the cash flow in one period is a deterministic function of the cash flow in the previous period plus a random variable that is independent of what has happened in all previous periods.

2.2 Problems with the Use of a Risk-adjusted Discount Rate – Examples

In many situations it is not reasonable to assume that the conditions mentioned in the previous section are satisfied. We shall show by three project examples what happens when these conditions are not fulfilled. The examples are stylised to visualise the effect of risk-adjusting the discount rate. However, their basic features are not unrealistic.

The cost of capital is assumed to be 12 %, and we assume that the relevant risk-adjusted discount rate has been found (using for example CAPM type arguments) to be 13 % for all three projects. All amounts are in millions of dollars.

Example 1:

We invest 98 in year 0 in building a VDSL access network. There is uncertainty as to whether the revenue from subscribers will commence in year 1 or 2. With probability 1/2 we receive a net income of 13 in year 1. In the years 2, 3, … we receive with probability 1 a yearly net income of 13. The uncertainty is thus whether the income stream starts in year 1 or 2 (see Figure 1 where the income that occurs with probability 1/2 is hatched).

Situation A – Net income 13 in year 1:

The NPV becomes $(13/1.12) / (1 - 1/1.12) - 98 = 10.3$.

Situation B – Net income 0 in year 1:

The NPV becomes $(13/1.12^2) / (1 - 1/1.12) - 98 = -1.3$.

Expected NPV: $1/2 \times 10.3 + 1/2 \times (-1.3) = 4.5$

Expected NPV with risk-adjusted discount rate:
$1/2 \times (13/1.13) / (1 - 1/1.13) + 1/2 \times (13/1.13^2) / (1 - 1/1.13) - 98 = -3.8$.

The method thus indicates that the project should be rejected.

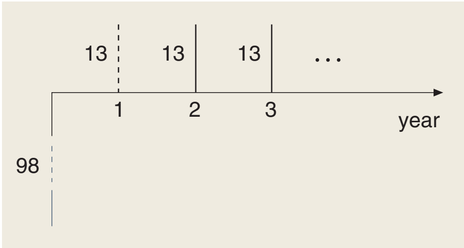


Figure 1 Cash flow in Example 1

Example 2:
We invest 63 in building an ADSL access network in year 0. There is uncertainty as to whether the ADSL technology will generate revenue for 4 or 5 years. With probability 1 we receive a net income of 20 in the years 1, ..., 4. With probability 1/2 we receive a net income of 20 in year 5. The uncertainty is thus whether we receive an income in year 5 (see Figure 2 where again the income that occurs with probability 1/2 is hatched).

Situation A – Net income 20 in year 5:
The NPV is $(20/1.12) \times (1 - (1 - 1/1.12)^5 / (1 - 1/1.12) - 63 = 9.1$.

Situation B – Net income 0 in year 5:
The NPV is $(20/1.12) \times (1 - (1 - 1/1.12)^4 / (1 - 1/1.12) - 63 = -2.3$.

Expected NPV:
$1/2 \times 9.1 + 1/2 \times (-2.3) = 3.4$

Expected NPV with risk-adjusted discount rate:
$1/2 \times (20/1.13) \times (1 - (1 - 1/1.13)^5 / (1 \times 1/1.13) + 1/2 \times (20/1.13) \times (1 - (1 - 1/1.13)^4 / (1 - 1/1.13) - 63 = 1.9$.

The method thus indicates that Example 2 should be accepted.

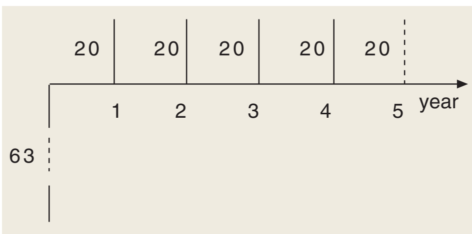


Figure 2 Cash flow in Example 2

Example 3:
We invest 98 in a VDSL access network in year 0. There is uncertainty as to whether the revenue from subscribers will commence in year 1 or 2. With probability 1/2 we receive a net income of 13 in year 1. In year 2 we sell the network to another company for $13 / (1 - 1/1.12) = 121.3$ (which is the NPV at capital cost of the incomes received with certainty in Example 1 from year 2 and onwards; see Figure 3). We see that Example 3 is financially equivalent to Example 1. We have only replaced the income in year 2 with the NPV at capital cost of the incomes we receive with certainty from year 2 and onwards.

Situation A – Net income 13 in year 1:
The NPV becomes $13/1.12 + 121.3/1.12^2 - 98 = 10.3$.

Situation B – Net income 0 in year 1:
The NPV becomes $121.3/1.12^2 - 98 = -1.3$.

Expected NPV:
$1/2 \times 10.3 + 1/2 \times (-1.3) = 4.5$.

Expected NPV with risk-adjusted discount rate:
$1/2 \times 13/1.13 + 121.3/1.13^2 - 98 = 2.8$.

The method thus indicates that Example 3 should be accepted.

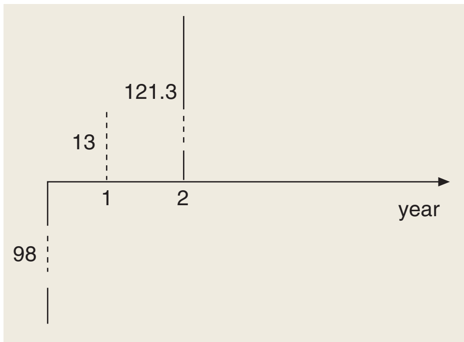


Figure 3 Cash flow in Example 3

These results are not reasonable. We see that the projects in Example 1 and Example 3 are financially equivalent. The criterion for whether they should be accepted or rejected should therefore be the same. Either should both be accepted or both be rejected. However, the expected NPV with risk-adjusted discount rate is negative for Example 1 and positive for Example 3. The NPV method with risk-adjusted discount rate thus implies that the project in Example 1 should be rejected whilst the project in Example 3 should be accepted. Furthermore, when the discount rate is set to capital cost, the project in Example 1 gives a higher NPV than the project in Example 2 for all statistical outcomes. Common sense thus implies that if the project in Example 2 is accepted, then the project in Example 1 should be accepted. The NPV method with risk-adjusted discount rate implies, however, that the project in Example 1 should be rejected whilst the project in Example 2 should be accepted.

The examples demonstrate that any method based on risk-adjusting the discount rate is unable to differentiate correctly between different risk structures. By risk-adjusting the discount rate one tries to satisfy two masters. The discount rate should reflect

- Capital cost (i.e. the rate of return which can be achieved by alternative application of the money which are tied up/released in the project over time). This has nothing to do with the risk of the project under consideration;
- The risk associated with the project

These two requirements are fundamentally different, and it is difficult to satisfy both by adjusting the discount rate. The main reason to look for an alternative approach is to be able to differentiate between capital cost and risk.

2.3 Risk-adjusted Expected Net Present Value

We shall now describe our suggested approach. The expected NPV is calculated using capital cost w as discount rate. The capital cost should reflect the average expected rate of return for alternative projects in the company and has nothing to do with the risk associated with the project under evaluation.

Then an amount is subtracted from the expected NPV. This amount adjusts the Expected NPV because of the risk of the project. The suggested measure of utility of a project is

$$\text{Expected NPV} - \alpha \times \text{risk} \qquad (2.3.1)$$

where α is a parameter to be determined. In financial theory the risk is traditionally expressed through the standard deviation of the NPV. This is a reasonable measure for portfolios of securities. Such portfolios consist of a number of different securities, and the return is the sum of the returns of the individual securities. The central limit theorem in probability theory indicates that this sum is approximately normally distributed. Risk should reflect the down side of the distribution of the NPV. However, the normal distribution is symmetric, so the standard deviation expresses the down side as well as the up side of the distribution. For individual investment projects in a company the assumption that the NPV has a symmetric probability distribution is obviously unreasonable. The NPV will often have a skewed distribution. As the standard deviation also reflects the up side of the distribution, we may have a large standard deviation even if the risk of the project is small.

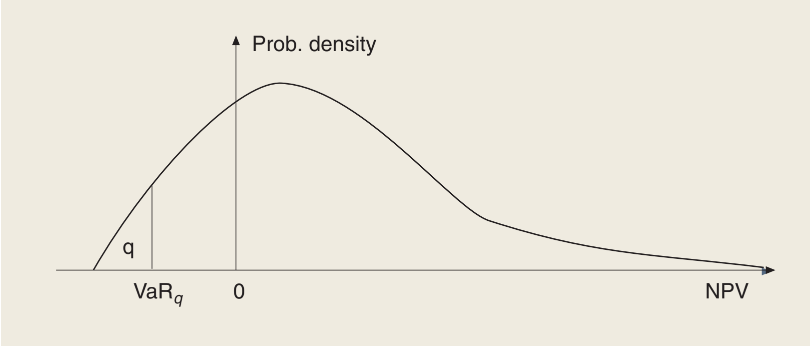


Figure 4 Value-at-Risk

Therefore we choose to measure risk by using the well-known concept *Value-at-Risk at q (VaR_q)*. VaR_q is defined as the q-point in the probability distribution of the NPV (see Figure 4).

q is usually chosen to be 1 % or 5 %. The suggested RAENPV then becomes

$$\text{Expected NPV} - \alpha \times (-\text{VaR}_q) \qquad (2.3.2)$$

where the NPV is calculated using the capital cost w as discount rate and where the parameter α reflects how the company (or its shareholders) weigh expected profit versus risk. We shall see later that the term $-\alpha \times (-\text{VaR}_q)$ can be interpreted as the insurance premium the company is willing to pay in order to be guaranteed against a negative NPV.

We shall now show how we determine α. For a perfect security market which contains a risk-free asset it is established knowledge that the substitution rate γ (price of risk) between expected return and risk (measured by the standard deviation of the return) in equilibrium is the same for all investors independent of their willingness to take risk (see e.g. [1] chapter 6). The value of γ can be obtained from stock exchange data and is not specific for a particular industrial sector. So for any investor an increment $\Delta\sigma$ in risk needs to be compensated by an increment of $\Delta\sigma/\gamma$ in expected return. This is shown in [1] for investments in a portfolio of securities where the portfolio is sold after one period, but it seems reasonable to use the same trade-off for multi-period projects. It is reasonable to measure the utility of an investment in a portfolio of securities by $E - \gamma\sigma$ where E is expected return and σ is the standard deviation of the return.

Figure 5 shows the classical picture where the dots represent securities in a diagram with the standard deviation of the return along the horizontal axis and the expected return along the vertical axis. The lower curve – the opportunity set border curve – limits all possible portfolios of securities. The straight line which starts at the risk-free asset and is tangent to the opportunity set border curve is the so-called *capital budget line*.

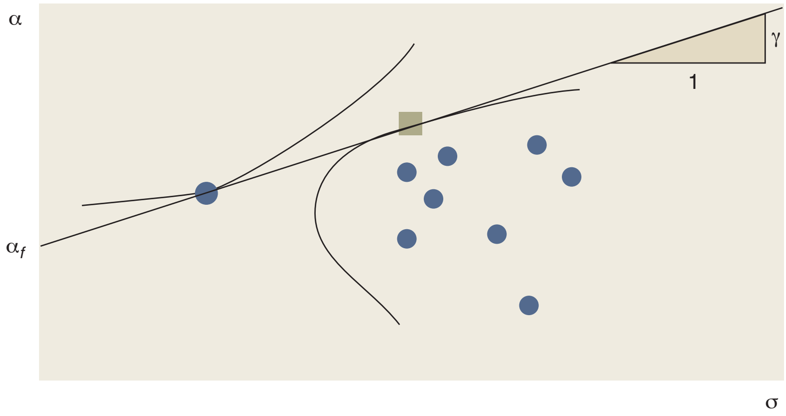


Figure 5 The capital budget line

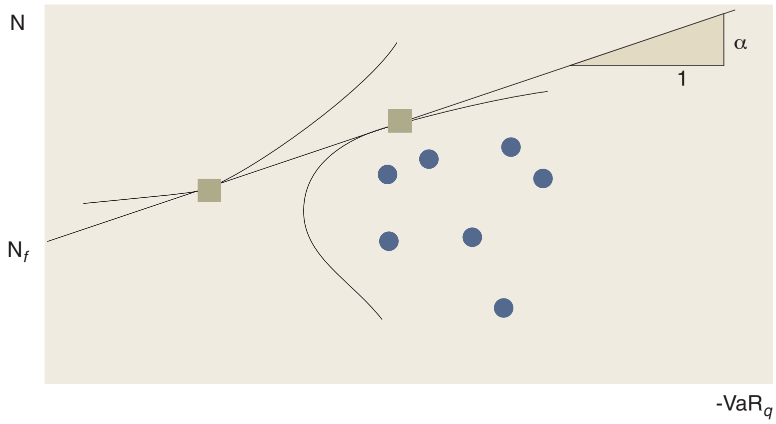
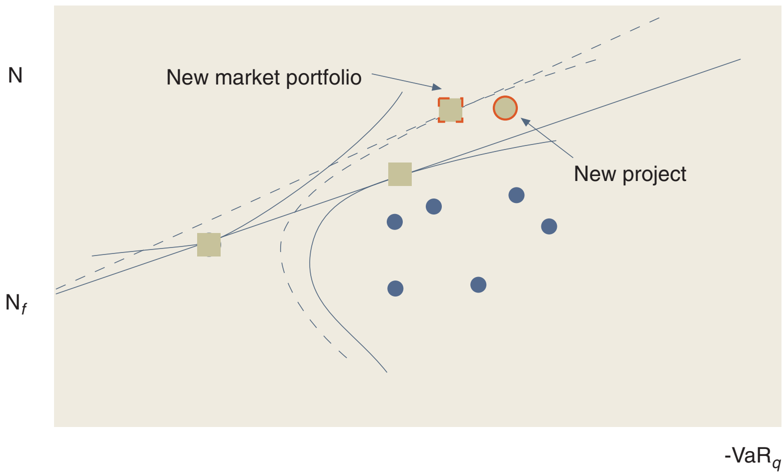


Figure 6 Transformed capital budget line

Figure 7 Project with higher risk-adjusted NPV than the market portfolio



The point where the capital budget line touches the opportunity set border curve represents the market portfolio and is marked by a small square in Figure 5. The upper curve in Figure 5 represents an indifference curve for an investor. The point where this curve touches the capital budget line, and which is marked by a small circle, represents the combination of the market portfolio and the risk-free asset which is considered to be optimal for this particular investor.

Figure 5 can be transformed to Figure 6 where we have replaced the standard deviation of the return by –Value-at-Risk for the NPV based on a discount rate r and the expected return by the expected NPV.

The rate of increase γ in Figure 5 has here been transformed to the rate of increase α in Figure 6. It can be shown (see the Appendix) that α **is independent of which discount rate** $\boldsymbol{r}$ **we apply**. We note further that the indifference curve for any investor has rate of increase equal to α at the point where it touches the transformed capital budget line. That is, α represents the optimal weighting between expected NPV and –Value-at-Risk. As mentioned earlier, the rate of increase γ can be found from capital market data. The formula for calculating α from γ is derived in the Appendix and is

$$\alpha = \frac{\gamma}{n_q - \gamma} \tag{2.3.3}$$

where n_q is the $(1 - q)$-point in the standardised normal distribution.

Since the market portfolio can be regarded as a yardstick, it is of interest to calculate the RAENPV for the market portfolio. In the Appendix it is shown that it can be expressed as

$$\frac{i - r}{(1 + r)(1 - \gamma / n_q)} \tag{2.3.4}$$

where i is the internal rate of return for the market portfolio.

A project should then be accepted if the RAENPV is greater than this. That will be in the interest of the shareholders. The project can then be represented by a point in Figure 6 that lies above the transformed capital market line. If the project is considered as a new investment possibility in the capital market, a larger opportunity set, a new transformed capital market line, and hence an improved market portfolio would arise. In Figure 7 the new opportunity set, capital market line, and market portfolio are indicated by broken lines. The new market portfolio would necessarily include the project.

An investment possibility represented by a point underneath the transformed capital market line might still improve the revised market portfolio and thus be a profitable investment. This would, however, be rather difficult to ascertain in general. We therefore propose to be conservative and leave to the planner the decision whether a project with a negative RAENPV should be accepted or not.

2.4 Analysis of the Examples Using Risk-adjusted Net Present Value

We shall now apply our criterion on the three examples described earlier where we assume that a has been estimated to be 0.2 and that the RAENPV of the market portfolio is very small.

In all three examples VaR_q at capital cost is the same whether q is 1 % or 5 % since the probability distributions of the NPV are different from zero for two values only, and for those values the probabilities are equal to 50 %.

Example 1:
The expected NPV and VaR_q with discount rate equal to capital cost are 4.5 and –1.3 respectively. The RAENPV becomes 4.5 – 0.2 × 1.3 = 4.24. The project should therefore be accepted.

Example 2:
The expected NPV and VaR_q with discount rate equal to capital cost are 3.4 and –2.3 respectively. The RAENPV becomes 3.4 – 0.2 × 2.3 = 2.94. The project should therefore be accepted. We see that the RAENPV is less than for the project in Example 1, which is reasonable since the project in Example 1 has a higher NPV at capital cost than the project in Example 2 under both statistical outcomes.

Example 3:
The expected NPV and VaR_q with discount rate equal to capital cost are 4.5 and –1.3 respectively. The RAENPV becomes 4.5 – 0.2 × 1.3 = 4.24. The project should therefore be accepted. The RAENPV is the same as for the project in Example 1, which is logical since the two projects are financially equivalent.

2.5 Establishing the necessary Data for Evaluation of a Single Project

The cash flow of a project is considered as a stochastic process. For every realisation of the process the NPV with capital cost as the discount rate can be calculated. This NPV is thus a random variable, and we propose the following procedure for estimating its probability distribution.

1 Establish a finite set of alternative future scenarios and assign a subjective probability of occurrence for each scenario.

2 For each scenario establish a pessimistic, an optimistic and a most probable cash flow for the project and calculate the NPV for each of those three realisations.

3 For each scenario specify an interval around the most probable realisation in which the NPV supposedly will lie with probability 1/2.

4 Fit a Beta distribution to the NPV for each scenario. Based on the mixture of these Beta distributions using the scenario probabilities, calculate the RAENPV.

Based on the resulting probability distribution of the NPV N one can calculate project characteristics like:

- Expected NPV ($\mathrm{E}N$)
- Value at risk at q ($\mathrm{VaR}_q(N)$)
- Standard deviation ($\sigma(N)$)
- Risk-adjusted ENPV ($\mathrm{E}N - \alpha(-\mathrm{VaR}_p(N))$)

Example 4:
We assume a single scenario. Based on a detailed analysis of the elements contributing to the cash flow the pessimistic and the optimistic estimates of the NPV for this scenario have been evaluated to be –400 and 300 respectively, and the most probable NPV is evaluated to be –100. The NPV is assumed to lie in the interval (-200, 0) with probability 1/2. (All figures are in mill. USD.)

The Beta distribution fit gives an expected NPV equal to –82.5 with standard deviation 134.4. The probability that the NPV is negative becomes 0.72 and $\mathrm{VaR}_{5\%}$ becomes –296. If α = .2, the RAENPV becomes –82.5 – .2 × 296 = –141.7.

3 Evaluation of Project Portfolios

3.1 Project Portfolio Alternatives

We shall consider the situation where we have a series of mutually exclusive portfolio alternatives. If we have n projects, there are in principle up to 2^n portfolio alternatives. In practice the number of alternatives will be much lower. Typically, some projects can be started only if some other projects are also started. Similarly there may be projects that cannot be part of the same portfolio. The planner must specify such restrictions, and a planning tool must facilitate the specification of such restrictions.

3.2 Scenarios

We can specify scenarios for project portfolios in two alternative ways:

- We can specify scenario alternatives across all projects.
- We can specify scenario alternatives for each individual project.

The two ways are in principle equivalent. If we for each project p specify the mutually exclusive scenarios $S_p^1, \mathrm{K}, S_p^{k(p)}$, we can define $\prod k(p)$ scenarios of the form $(S_1^{i_1} \cap S_2^{i_2} \cap \mathrm{L} \cap S_n^{i_n})$ which run across projects. We shall in the sequel assume that the scenarios are defined across the projects.

3.3 Evaluation of a Portfolio of Projects

The evaluation of a portfolio is done by a straightforward generalisation of the approach for a single project. Based on the probability distributions of the NPVs of the individual projects and the correlations between them we can calculate the same characteristics as for a single project such as expected NPV, value-at-risk, standard deviation, and RAENPV.

In general one proceeds as follows:

- Establish a set of mutually exclusive scenario alternatives. Which of these scenarios that occurs should have a significant influence on the NPV of the portfolio. Each scenario is assigned a subjective probability of occurrence.

- For each scenario establish a pessimistic, an optimistic and a most probable realization of each project in the portfolio and calculate the NPV for each of these realizations.

- For each scenario and each project in the portfolio specify an interval around the most probable realisation where it is assumed that the NPV will lie with probability 1/2.

- For each scenario specify a matrix of correlation coefficients between the NPVs of the individual projects in the portfolio.

- For each scenario calculate the expectation and the variance of the NPV of the portfolio, and approximate the distribution of the NPV by a suitable Beta distribution.

- Based on the mix of these Beta distributions over the scenarios, calculate the wanted characteristics of the portfolio's NPV.

3.4 Establishing Covariances

3.4.1 General

Establishing correlation coefficients (or equivalently, covariances) between the NPVs of all pairs of projects in a portfolio is not trivial. CAPM (see [1]) has an elegant solution to this for portfolios of securities which are registered on a stock exchange. Here it is only necessary to establish the covariance between the portfolio and the market portfolio. Unfortunately, the necessary conditions are in general unrealistic, even for portfolios of securities. For project portfolios these conditions cannot be justified at all. However, in this case the number of covariances to be estimated is considerably lower. In a security market the investors can choose between hundreds of securities so that there will be tens of thousands of covariances. A portfolio of investment projects in a company will normally consist of between 5 and 10 projects so that the number of covariances will be less than 50. We shall briefly discuss two approaches to estimation of the covariance between two NPVs, namely *direct estimation* and the *use of independent explanatory variables*.

However, we should make note of the fact that the RAENPV measure is approximately *additive* in the sense that the RAENPV of a union of two projects is approximately equal to the sum of the RAENPV of the individual projects. The expected NPV is obviously additive. The same is true for VaR_q when $q = 0$ %, and the two projects are not totally negatively correlated. So for $q = 1$ %, the approximation may not be too misleading. For preliminary analyses we may thus skip the correlation calculations altogether.

3.4.2 Direct Estimation

Here the planner estimates the covariances, or rather the correlation coefficients, based on experience and sound judgement. In order to facilitate this a planning tool should enable the planner to choose from a selection of preset correlation coefficients as suggested in Table 1.

Example 5

Suppose that we have three projects A, B and C with NPVs NA, NB and NC. We estimate the correlation coefficients for all pairs of NPVs as given in Table 2.

Not every table set up in this manner gives rise to valid correlation matrices. A necessary, but not sufficient, condition is that the matrix is positive semidefinite. Based on the distributions of the NPVs it is also possible to establish lower and upper bounds on the individual correlation coefficients. This, however, is computationally demanding and probably not worth the effort. These bounds will very seldom lie in the interval [–3/4,3/4], and total correlations (±T) will only

Table 1 Correlation codes

Description	Total negative	Large negative	Medium negative	Small negative	None	Small positive	Medium positive	Large positive	Total positive
Code	–T	–L	–M	–S	N	S	M	L	T
Correlation coefficient	–1	–3/4	–1/2	–1/4	0	1/4	1/2	3/4	1

be used when the NPVs are linearly dependent. Then the bounds will be –1 and +1 and thus be without value.

If the planner has set up a correlation matrix which is not positive semidefinite, the planning tool should be able to suggest a modified correlation matrix which is positive semidefinite.

3.4.3 Use of Independent Explanatory Variables

We assume that we can write the NPVs N_1 and N_2 for two investment projects 1 and 2 as functions of F independent random variables V_1, V_2, ..., V_F with known distributions:

$$N_p = N_p(V_1, V_2, ..., V_F) \text{ for } p = 1,2. \quad (3.4.1)$$

The V-s are explanatory variables which determine e.g. equipment costs, total markets, and market shares. Furthermore, we assume that the functions N_p can be approximated by generalised polynomials:

$$N_p(V_1, V_2, \mathrm{K}, V_F) \approx \sum_{i_1,\mathrm{K},i_F} \alpha^p_{i_1\mathrm{K}i_F} V_1^{i_1} \mathrm{K} V_F^{i_F} \quad (3.4.2)$$

where i_1, ..., i_F may take a finite number of integer values which are not necessarily positive. We are then able to calculate the covariance between N_1 and N_2:

$$\mathrm{cov}(N_1, N_2) = \mathrm{E}(N_1N_2) - \mathrm{E}N_1\mathrm{E}N_2 \quad (3.4.3)$$

where

$$\mathrm{E}(N_1N_2) = \sum_{\substack{i_1,\mathrm{K},i_F \\ j_1,\mathrm{K},j_F}} \alpha^1_{i_1\mathrm{K}i_F}\alpha^2_{j_1\mathrm{K}j_F} \mathrm{E}V_1^{i_1+j_1}\mathrm{K}\mathrm{E}V_F^{i_F+j_F} \quad (3.4.4)$$

and

$$\mathrm{E}N_1\mathrm{E}N_2 = \sum_{\substack{i_1,\mathrm{K},i_F \\ j_1,\mathrm{K},j_F}} \alpha^1_{i_1\mathrm{K}i_F}\alpha^2_{j_1\mathrm{K}j_F} \mathrm{E}V_1^{i_1}\mathrm{E}V_1^{j_1}\mathrm{K}\mathrm{E}V_F^{i_F}\mathrm{E}V_F^{j_F} \quad (3.4.5)$$

such that

$$\mathrm{cov}(N_1, N_2) = \sum_{\substack{i_1,\mathrm{K},i_F \\ j_1,\mathrm{K},j_F}} \alpha^1_{i_1\mathrm{K}i_F}\alpha^2_{j_1\mathrm{K}j_F} (\mathrm{E}V_1^{i_1+j_1}\mathrm{K}\mathrm{E}V_F^{i_F+j_F} - \mathrm{E}V_1^{i_1}\mathrm{E}V_1^{j_1}\mathrm{K}\mathrm{E}V_F^{i_F}\mathrm{E}V_F^{j_F}) \quad (3.4.6)$$

We assume furthermore that V_f is normalized to lie between 0 and 1 and is Beta distributed with parameters μ_f and ν_f. Then

$$\mathrm{E}V_f^i = \frac{\Gamma(\mu_f + \nu_f)\Gamma(\mu_f + i)}{\Gamma(\nu_f)\Gamma(\mu_f + \nu_f + i)} \quad (3.4.7)$$

Table 2 Specification of correlations

	N_B	N_C
N_A	M	–L
N_B		–S

Using (3.4.6) and (3.4.7) we can find an expression for cov(N_1, N_2). Analogously we may find formulas for var N_p for relevant portfolios p.

Example 6

$N_1 = 10V_1 + V_2$
$N_2 = 2V_1 - 3V_2$

where V_1 is Beta distributed over (0,1) with parameters μ_{n_1} = 3 and ν_{n_1} = 6, whilst V_2 is Beta distributed over (0,1) with parameters μ_{n_2} = 4 and ν_{n_2} = 2, and where V_1 and V_2 are independent.

Then

$$\mathrm{E}V_1 = \frac{6}{3+6} = \frac{2}{3}, \mathrm{E}V_2 = \frac{2}{2+4} = \frac{1}{3}$$

$$\mathrm{var}V_1 = \frac{3 \times 6}{(3+6)^2(3+6+1)} = \frac{1}{45},$$
$$\mathrm{var}V_2 = \frac{4 \times 2}{(4+2)^2(4+2+1)} = \frac{2}{63},$$

$$\mathrm{cov}(N_1, N_2) = \mathrm{cov}(10V_1 + V_2, 2V_1 - 3V_2)$$
$$= 20\mathrm{var}V_1 - 3\mathrm{var}V_2 = \frac{22}{63}$$

$$\mathrm{E}N_1 = 10\mathrm{E}V_1 + \mathrm{E}V_2 = 7,$$
$$\mathrm{E}N_2 = 2\mathrm{E}V_1 - 3\mathrm{E}V_2 = \frac{1}{3}$$

$$\mathrm{var}N_1 = 100\mathrm{var}V_1 + \mathrm{var}V_2 = \frac{142}{63},$$
$$\mathrm{var}N_2 = 4\mathrm{var}V_1 + 9\mathrm{var}V_2 = \frac{118}{315}.$$

We may approximate the distributions of N_1 and N_2 by Beta distributions. The intervals are determined by:

$a_{N_1} = 0$, $b_{N_1} = 11$, $a_{N_2} = -3$, $b_{N_2} = 2$.

In order to find the parameters we normalize N_1 and N_2:

$$\tilde{N}_1 = \frac{N_1}{11}, \mathrm{E}\tilde{N}_1 = \frac{7}{11}, \mathrm{var}\tilde{N}_1 = \frac{142}{7623},$$
$$\tilde{N}_2 = \frac{N_2 + 3}{5}, \mathrm{E}\tilde{N}_2 = \frac{2}{3}, \mathrm{var}\tilde{N}_2 = \frac{118}{7875}.$$

This gives

$$\mu_{N1} = \frac{(7/11)(1-7/11)^2}{142/7623} - (1-7/11) = 4.15,$$

$$\nu_{N1} = \frac{(7/11)^2(1-7/11)}{142/7623} - 7/11 = 7.27$$

$$\mu_{N2} = \frac{(2/3)(1-2/3)^2}{118/7875} - (1-2/3) = 4.61,$$

$$\nu_{N2} = \frac{(2/3)^2(1-2/3)}{118/7875} - 1/3 = 9.55.$$

4 Extension to Options

4.1 A Single Project

We shall first focus on the analysis of a single project. We consider a Project *A* that we may start at some future time *T*. Between current time T_0 and time *T* we may receive information which can influence our decision whether to launch Project *A* or not. If, for example, this project competes with an alternative Project *B* with launching time T_0, we must consider the value of the option of not starting Project *A* that we will have at time *T*. If the information we receive during the time interval $[T_0,T]$ implies that the probability that the project will be profitable is unacceptably low we will at time *T* exercise the option of not starting the project. This should in general give a higher utility of the project than what we would get if we base the analysis merely on estimates of the cash flow of the project as seen at time T_0. What is required, however, is the probability distribution of the additional information as seen at time T_0.

Here we shall limit ourselves to describing a discrete options model where the additional information that we receive in $[T_0,T]$ relevant for the project in question is an adjustment of the probability of occurrence of the individual scenarios. We thus assume that we do not receive any additional information that is sufficiently significant to warrant a change in the Beta distributions for the individual scenarios. This may seem to be a severe limitation. It is, however, mitigated somewhat by the possibility of splitting a scenario where we would like to change the NPV distribution into two or more subscenarios.

We define λ^m to be the probability at time T_0 that we at time *T* are in *situation m* amongst *M* alternative situations. Situation *m* is characterized by a probability distribution over the different scenarios. The RAENPV for scenario *i* is n^i. All n^i are assumed to be known. So the uncertainty at time *T* is reduced to the question as to which situation that will occur. Let $\underline{n}$ be the RAENPV of the market portfolio. The overall RAENPV *n* for the project becomes

$$n = \sum_m \lambda^m n^m. \tag{4.1.2}$$

4.2 Portfolios

We now proceed to the analysis of portfolios of projects where each project has a starting date some time in the future and where information received before the starting date may indicate that it is not profitable to start the project.

Again we shall limit ourselves to describing a discrete options model where the additional information that we receive before the possible start of a project in the portfolio is an adjustment of the probability of occurrence of the individual scenarios. We enumerate all the projects according to increasing starting time: P_1, P_2, ..., and we want to decide whether to start project P_1. We shall approach this problem using dynamic programming. We let D_p denote a partial portfolio of projects amongst P_1, ..., P_p, and we denote by *m'* a situation at the starting time for project P_{p+1}. We assume that we for all such partial portfolios D_p and all *m'* know the partial portfolio $D_p'(m')$ amongst the projects P_{p+1}, P_{p+2} ... which together with D_p yield the portfolio $D_p \cup D_p'(m')$ with highest RAENPV.

We now assume that we have decided on a partial portfolio D_{p-1} amongst the projects P_1, ..., P_{p-1}, that we are in situation *m*, and that we face the decision whether to start project P_p. We now consider two cases[1]:

(i) We start project P_p. Then we form the partial portfolio $D_p = D_{p-1} \cup P_p$. The probability of being in situation *m'* at the starting time for project P_{p+1} is assumed to be known and equal to $\lambda_p^{mm'}$. The portfolio $D_p \cup D_p'(m')$ with the highest RAENPV $n(D_p, m')$ in situation *m'* is known, and the RAENPV becomes

$$\sum_{m'} \lambda_p^{mm'} n(D_p, m') \tag{4.2.1}$$

(ii) We do not start project P_p. The portfolio $D_{p-1} \cup D_p'(m')$ with highest RAENPV $n(D_{p-1}, m')$ in situation *m'* is known, and the RAENPV becomes

$$\sum_{m'} \lambda_p^{mm'} n(D_{p-1}, m') \tag{4.2.2}$$

[1] *Note that in many situations we cannot choose whether we should start project P_p or not. It may happen that P_p either is not compatible with D_{p-1} or that P_p is a necessary consequence of D_{p-1}. This contributes to a reduction of the number of possible portfolios and thus the time required for the necessary calculations.*

If (4.2.1) is larger than (4.2.2), we choose to start project P_p. If (4.2.1) is less than or equal to (4.2.2) we choose not to start project P_p. For every partial portfolio D_{p-1} of the projects $P_1, ..., P_{p-1}$, and every situation m at the start time for project P_p, we thus know the partial portfolio $D_{p-1}'(m)$ of the projects P_p, P_{p+1} ..., which together with D_{p-1} yield the portfolio $D_{p-1} \cup D_{p-1}'(m)$ with highest RAENPV $n(D_{p-1}, m)$.

This then forms the basis for a dynamic programming algorithm for finding the project portfolio with the highest RAENPV. The input we need in addition to what we have discussed earlier is the Markov chain matrices $\Lambda_p = (\lambda_p^{mm'})$ where $\lambda_p^{mm'}$ is the probability of being in situation m' at the starting time of project P_{p+1} given that we are in situation m at the starting time of project P_p.

To get going we start with the project P_n with the latest starting time and then work our way backwards in time:

For every partial portfolio D_{n-1} of projects from $P_1, ..., P_{n-1}$, and every situation m at the starting time for project P_n we consider two cases:

(i) We start project P_n, form the portfolio $D_n = D_{n-1} \cup P_n$ and calculate the RAENPV $n(D_n)$.

(ii) We do not start project P_n, put $D_n = D_{n-1}$ and calculate the RAENPV $n(D_n)$.

If the RAENPV we calculated in (i) is greater than the RAENPV we calculated in (ii), we choose to start project P_n. If the RAENPV we calculated in (i) is less than or equal to the RAENPV we calculated in (ii), we choose not to start project P_n. For every partial portfolio D_{n-1} from the projects $P_1, ..., P_{n-1}$, and every situation m at the starting time for project P_n we thus know whether it is profitable or not to start project P_n and can therefore for every situation m establish the extension of the partial portfolio D_{n-1} to a complete portfolio with the highest RAENPV. The set of relevant situations may of course vary from one project starting point to the next.

Example 7

We assume that we have five situations $S_1, ..., S_5$, and three projects P_1, P_2 and P_3 with startup times $t_1 < t_2 < t_3$. At t_1 we are in situation S_1. We assume that the transitions between the situations are as depicted in Figure 8.

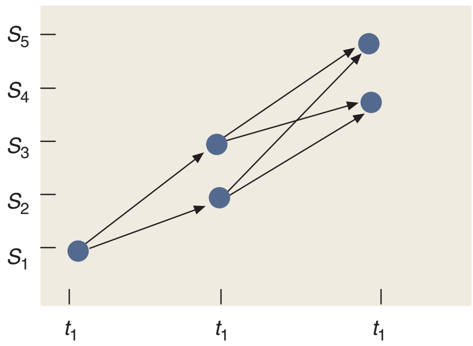


Figure 8 Situation transitions

The transition probabilities are given in Table 3.

Table 3 Transition probabilities

	S_2	S_3	S_4	S_5
S_1	1/2	1/2		
S_2			2/3	1/3
S_3			1/3	2/3

The RANPVs of the three projects in the relevant situations are given in Table 4.

Table 4 RANPVs for the individual projects

	P_1	P_2	P_3
S_1	1		
S_2		–5/4	
S_3		1/4	
S_4			5/2
S_5			–1/2

We assume further that P_3 cannot be started if P_2 is not started. Thus the possible portfolios are Ø, P_1, P_2, $P_1 \cup P_2$, $P_2 \cup P_3$, $P_1 \cup P_2 \cup P_3$. We want to know whether we shall start project P_1 at time t_1. We also make use of the approximate additivity of RAENPV mentioned in 3.4.1. We shall fill in Table 5 where the entries are to be interpreted as RAENPVs disregarding contributions from projects started earlier in time, and optimal decisions are made at later points in time.

From Table 4 we see that we at time t_3 start project P_3 in situation S_4 whilst we do not start project P_3 in situation S_5. Thus the rightmost column in Table 5 is easily established.

If we do not start P_2, the tail RAENPV becomes 0.

If we start P_2 in S_2, the expected tail RAENPV becomes –5/4 + 2/3 × 5/2 = 5/12.

If we start P_2 in S_3, the expected tail RAENPV becomes 1/4 + 1/3 × 5/2 = 13/12.

If we do not start P_1, the expected RAENPV becomes 1/2 × 5/12 + 1/2 × 13/12 = 3/4.

If we start P_1, the expected RAENPV becomes 1 + 1/2 × 5/12 + 1/2 × 13/12 = 7/4.

Thus we conclude that project P_1 should be started.

Table 5 RANPVs for the tail portfolios

	P_1	P_2	P_3
S_1	7/4		
S_2		5/12	
S_3		13/12	
S_4			5/2
S_5			0

5 Conclusion

We have argued that it makes more sense to evaluate portfolios based on a utility measure that balances expected net present value at capital cost against value-at-risk, rather than to use a risk-adjusted discount rate. The concepts used are fairly simple, and the approach can readily be implemented in a spreadsheet tool that calculates the total utility of all relevant portfolio candidates and ranks them according to decreasing total utility. We believe that the output of such a tool will be useful for the planners responsible for portfolio analysis and investment recommendations.

Appendix: Risk-Adjusted Expected NPV – Detailed Analysis

A1 The Market Portfolio

We consider a suitable market portfolio M and operate with a fixed period length which we for simplicity set equal to one year. At the beginning of the year we buy for one dollar worth of the market portfolio and sell it at the end of the year for W dollars. The rate of return A is then given by

$$A = \frac{W-1}{1} = W - 1. \tag{A1}$$

The NPV N_r^M with discount factor r then becomes

$$N_r^M = \frac{W}{1+r} - 1 = \frac{A-r}{1+r}. \tag{A2}$$

with expected value

$$\mathrm{E}N_r^M = \frac{\mathrm{E}A - r}{1+r}. \tag{A3}$$

We thus have

$$\mathrm{E}A = (1+r)\mathrm{E}N_r^M + r. \tag{A4}$$

Let a_q be the q-point in the distribution of A. Then

$$q = \mathrm{P}(A \le a_q) = \mathrm{P}\left(\frac{A-r}{1+r} \le \frac{a_q - r}{1+r}\right) = \mathrm{P}\left(N_r^M \le \frac{a_q - r}{1+r}\right) \tag{A5}$$

such that

$$\mathrm{VaR}_q(N_r^M) = \frac{a_q - r}{1+r}. \tag{A6}$$

We can estimate a_q based on the empirical distribution of A by using the q-point in this distribution. Since the return of the market portfolio is a weighted sum of the returns of the individual securities that constitute the portfolio we may alternatively permit ourselves to assume that A is normally distributed N(EA, σ_A). Then we can establish the following relation

$$q = \mathrm{P}(A \text{ "} a_q) = \mathrm{P}\left(\frac{A - \mathrm{E}A}{\sigma_A} \le \frac{a_q - \mathrm{E}A}{\sigma_A}\right) \tag{A7}$$

which gives

$$\frac{a_q - \mathrm{E}A}{\sigma_A} = -n_q \tag{A8}$$

where $-n_q$ is the q-point in the standardized normal distribution. (A8) can be written as

$$a_q = \mathrm{E}A - n_q\sigma_A. \tag{A9}$$

This gives further

$$\mathrm{VaR}_q(N_r^M) = \frac{\mathrm{E}A - n_q\sigma_A - r}{1+r} = \mathrm{E}N_r^M - \frac{n_q\sigma_A}{1+r} \tag{A10}$$

which gives the following expression for σ_A:

$$\sigma_A = \frac{1+r}{n_q}\left(\mathrm{E}N_r^M - \mathrm{VaR}_q(N_r^M)\right) \tag{A11}$$

The capital market line is given by:

$$\mathrm{E}A = f + \gamma\sigma_A \tag{A12}$$

where f is the risk free interest rate. γ reflects how the market portfolio balances expected return against risk[2)].

[2)] *A new investment possibility that lies above the capital market line will be included in the market portfolio and thus alter it. The investment will therefore be considered profitable in a perfect capital market. (An investment possibility below the capital market line could also be profitably included in the market portfolio, but this is considerably more difficult to ascertain.)*

From (A9) we see that

$$EA = a_q + n_q \sigma_A \quad \text{(A13)}$$

We assume that the market portfolio is not risk free, in other words that $a_q < f$. (A12) og (A13) then imply that the coefficient of substitution γ between return and risk is less than n_q. (A12) gives the following expression for γ:

$$\gamma = \frac{EA - f}{\sigma_A}. \quad \text{(A14)}$$

By substituting (A4) and (A11) into (A12) we get an analogous 'capital market line' which ties together EN_r^M and $VaR_q(N_r^M)$:

$$(1+r)EN_r^M + r = f + \gamma \frac{1+r}{n_q} \left(EN_r^m - VaR_q(N_r^M) \right) \quad \text{(A15)}$$

or

$$EN_r^M = \frac{(f-r)n_q}{(1+r)(n_q - \gamma)} + \alpha \left[-VaR_q(N_r^M) \right] \quad \text{(A16)}$$

where $\alpha = \frac{\gamma}{n_q - \gamma}$. This is meaningful since

$\gamma < n_q$.

α represents the balancing of expected NPV against risk. We observe that α is independent of r.

A2 The Profitability of a Project

We propose to measure the profitability of a project by balancing the expected NPV against the value-at-risk where α is used as the weighting coefficient. Thus we choose to let the company's willingness to take risk be the same as for the market portfolio. The expected NPV and the value at risk depend on the choice of discount rate. We suggest the use of the company's estimated average capital cost as a value for r. It represents the average return on investment within the company and is thus a good estimate of what the company may obtain through an alternative use of the money. (We are not dogmatic here. Other considerations may cause the choice of another discount factor.) Thus the profitability of a project P is measured by the risk-adjusted expected NPV (RAENPV) given by

$$EN_r - \alpha\,(-VaR_q(N_r)). \quad \text{(A17)}$$

The definition (2.3.2) together with (A16) gives that the RAENPV of the market portfolio is

$$\frac{f-r}{(1+r)(1-\gamma/n_q)}. \quad \text{(A18)}$$

References

1 Copeland, T E, Weston, J F. *Financial Theory and Corporate Policy*. Boston, Mass, Addison Wesley, 1992.

2 Mina, J, Xiao, J Y. *Return to Risk Metrics: The Evolution of a Standard*. New York, RiskMetrics, 2001.